CALDERÓN-ZYGMUND ESTIMATES

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ABSTRACT. In this note, we prove Calderón-Zygmund estimates in two ways.

1. MOTIVATION AND DEFINITIONS

In this note, we will prove the celebrated result of Calderón-Zygmund: for $1 < p < \infty$, there exists a constant $C = C(n, p) > 0$ such that

$$\|\nabla^2 u\|_{L_p} \leq C\|\Delta u\|_{L_p} \tag{1.1}$$

for all $u \in C_c^\infty$.

We first give a motivation for obtaining such estimate. Let us consider the Newtonian potential

$$\mathcal{N}[f](x) = \int_{\mathbb{R}^n} \Phi(y) f(x-y)\, dy$$

for $f \in C_c^\infty$. Here Φ is the fundamental solution of $-\Delta$ defined by

$$\Phi(x) = \begin{cases} \frac{1}{n(n-2)\omega_n} \frac{1}{|x|^{n-2}} & \text{if } n \geq 3, \\ -\frac{1}{2\pi} \ln |x| & \text{if } n = 2, \end{cases}$$

where ω_n is the volume of the unit ball in $\mathbb{R}^n$. For simplicity, we assume that $n \geq 3$. Since $f \in C_c^\infty$ and $\nabla\Phi$ is locally integrable, integration by part shows that

$$D_i[\mathcal{N} f](x) = -\frac{1}{n\omega_n} \int_{\mathbb{R}^n} \frac{y^i}{|y|^n} f(x-y)\, dy.$$

However, if we differentiate it again, we cannot pass the derivatives of f to the kernel since $|\nabla^2 \Gamma(x)|$ is not locally integrable.

To get an appropriate representation, choose a radial function $\zeta \in C_c^\infty$ satisfying $\zeta(0) = 1$. Then

$$\begin{aligned} D_{ij}(\mathcal{N}[f])(x) = \frac{1}{n\omega_n} &\int_{\mathbb{R}^n} \frac{y_i}{|y|^n} [f(x-y) - f(x)\zeta(y)]_{y_j}\, dy \\ &+ \frac{1}{n\omega_n} \int_{\mathbb{R}^n} \frac{y_i}{|y|^n} [f(x)\zeta(y)]_{y_j}\, dy. \end{aligned}$$

Observe that

$$\frac{1}{n\omega_n} \int_{\mathbb{R}^n} \frac{y^i}{|y|^n} [\zeta(y)]_{y_j}\, dy = \frac{1}{n}\delta_{ij}.$$

Indeed, choose $R > 0$ so that $\operatorname{supp} \zeta \subset B_R$. Then for any $\varepsilon > 0$, integration by parts gives

$$\frac{1}{n\omega_n} \int_{B_R \setminus B_\varepsilon} \frac{y_i}{|y|^n} [\zeta(y)]_{y_j}\, dy = -\frac{1}{n\omega_n} \int_{B_R \setminus B_\varepsilon} \partial_{y_j} \left(\frac{y_i}{|y|^n} \right) \zeta(y)\, dy$$

$$+\frac{1}{n\omega_n}\int_{\partial B_\varepsilon}\frac{y_i}{|y|^n}\zeta(y)\left(\frac{-y_j}{|y|}\right)\,d\sigma(y)$$
$$=-\frac{1}{n\omega_n}\int_{B_R\setminus B_\varepsilon}\left(\frac{\delta_{ij}|y|^2-ny_iy_j}{|y|^{n+2}}\right)\zeta(y)\,dy$$
$$+\frac{1}{n\omega_n}\int_{\partial B_\varepsilon}\frac{y_i}{|y|^n}\zeta(y)\left(\frac{-y_j}{|y|}\right)\,d\sigma(y).$$

Observe that the integrals on the right-hand side are zero if $i\neq j$ if we use the fact (1.3) and ζ is radial. If $i=j$, then

$$\frac{1}{n\omega_n}\int_{B_R\setminus B_\varepsilon}\frac{y_i}{|y|^n}[\zeta(y)]_{y_i}\,dy=-\frac{1}{n\omega_n}\int_{B_R\setminus B_\varepsilon}\left(\frac{|y|^2-ny_i^2}{|y|^{n+2}}\right)\zeta(y)\,dy$$
$$+\frac{1}{n\omega_n}\int_{\partial B_\varepsilon}\frac{y_i^2}{|y|^{n+1}}\zeta(y)\,d\sigma.$$

A change of variable shows that

$$\frac{1}{n\omega_n}\int_{B_R\setminus B_\varepsilon}\left(\frac{|y|^2-ny_i^2}{|y|^{n+2}}\right)\zeta(y)\,dy=\frac{1}{n\omega_n}\int_{B_R\setminus B_\varepsilon}\left(\frac{|y|^2-ny_1^2}{|y|^{n+2}}\right)\zeta(y)\,dy$$

for $1\le i\le n$. Similarly,

$$\frac{1}{n\omega_n}\int_{\partial B_\varepsilon}\frac{y_i^2}{|y|^{n+1}}\zeta(y)\,d\sigma=\frac{1}{n\omega_n}\int_{\partial B_\varepsilon}\frac{y_1^2}{|y|^{n+1}}\zeta(y)\,d\sigma$$

for $1\le i\le n$. Hence if we denote

$$A_\varepsilon=\frac{1}{n\omega_n}\int_{B_R\setminus B_\varepsilon}\frac{y_i}{|y|^n}[\zeta(y)]_{y_i}\,dy,$$

then

$$nA_\varepsilon=-\sum_{i=1}^n\frac{1}{n\omega_n}\int_{B_R\setminus B_\varepsilon}\left(\frac{n|y|^2-n|y|^2}{|y|^{n+2}}\right)\zeta(y)\,dy$$
$$+\frac{1}{n\omega_n}\int_{\partial B_\varepsilon}\frac{1}{|y|^{n-1}}\zeta(y)\,d\sigma$$
$$=\frac{1}{n\omega_n\varepsilon^{n-1}}\int_{\partial B_\varepsilon}\zeta(y)\,d\sigma(y).$$

Since $\sigma(\partial B_\varepsilon)=n\omega_n\varepsilon^{n-1}$, it follows from continuity of ζ and $\zeta(0)=1$ that

$$\lim_{\varepsilon\to 0+}nA_\varepsilon=1.$$

This implies that

$$\frac{1}{n\omega_n}\int_{\mathbb{R}^n}\frac{y_i}{|y|^n}[\zeta(y)]_{y_j}\,dy=\frac{1}{n}\delta_{ij}.$$

Since

$$|f(x-y)-f(x)\zeta(y)|\le|f(x-y)-f(x)+f(x)(1-\zeta(y))|\le C|y|,$$

it follows that

$$D_{ij}(\mathcal{N}[f])(x)=C_1(n)\int_{\mathbb{R}^n}K_{ij}(y)(f(x-y)-f(x)\zeta(y))\,dy-\frac{\delta_{ij}}{n}f(x)$$

converges, where

$$K_{ij}(y) = \frac{\delta_{ij}|y|^2 - ny_iy_j}{|y|^{n+2}}.$$

Since ζ is radial, then one can show that

$$\int_{|y|\geq\varepsilon} K_{ij}(y)\zeta(y)\,dy = 0.$$

Hence we obtain the following representation

$$D_{ij}(\mathcal{N}[f])(x) = C_1(n)\mathcal{K}_{ij}f(x) - \frac{\delta_{ij}}{n}f(x),$$

where

$$\mathcal{K}_{ij}f(x) = \lim_{r\to 0+}\int_{|y|\geq r} K_{ij}(y)f(x-y)\,dy.$$

This leads us to study the Calderón-Zygmund theory of singular integrals.

Definition 1.1. We say that a kernel $K : \mathbb{R}^n \setminus \{0\} \to \mathbb{C}$ is the *Calderón-Zygmund kernel* if

(1) there exists a constant $B > 0$ such that $|K(x)| \leq B|x|^{-n}$ for all $x \neq 0$;
(2) (Hörmander condition) there exists a constant $C > 0$ such that

$$\int_{|x|>2|y|} |K(x) - K(x-y)|\,dx \leq C$$

for all $y \neq 0$;
(3) (Cancellation property) For any $0 < r < s < \infty$, we have

$$\int_{r<|x|<s} K(x)\,dx = 0.$$

Remark. The Hörmander condition is related to the regularity condition on K. In fact, one can show that if $|\nabla K(x)| \leq B/|x|^{n+1}$ for some $B > 0$, then K satisfies the Hörmander condition. Indeed, by the fundamental theorem of calculus, we have

$$K(x-y) - K(x) = -\int_0^1 \nabla K(x-ty)\cdot y\,dt$$

Since $x - ty \in B_{|x|/2}(x)$, it follows that $|x-ty| \geq \frac{|x|}{2}$.

$$\begin{aligned}
\int_{|x|\geq 2|y|} |K(x) - K(x-y)|\,dx &= \int_{|x|\geq 2|y|}\left|\int_0^1 \nabla K(x-ty)\cdot y\,dt\right| dx \\
&\leq C\int_{|x|\geq 2|y|} |y|\frac{1}{|x|^{n+1}}\,dx \\
&= C|y|\int_{2|y|}^{\infty}\frac{1}{\rho^2}d\rho = C.
\end{aligned}$$

Proposition 1.2. *Let $f \in \mathcal{S}(\mathbb{R}^n)$ and let K be a Calderón-Zygmund kernel. Then the limit*

$$Tf(x) = \lim_{\varepsilon\to 0+}\int_{|x-y|>\varepsilon} K(x-y)f(y)\,dy$$

exists.

Proof. Note that for $0<\varepsilon<1$, it follows from the cancellation condition that

$$\int_{|y|>\varepsilon} K(y)f(x-y)\,dy = \int_{\varepsilon<|y|<1} K(y)[f(x-y)-f(x)]\,dy + \int_{|y|>1} K(y)f(x-y)\,dy.$$

By the mean value theorem, we have

$$|f(x-y)-f(x)| \le \|\nabla f\|_{L_\infty}|y|. \tag{1.2}$$

Since $K(y)$ is bounded for $|y|\ge 1$ and $f\in\mathcal{S}$, the second integral is well-defined. Also, by (1.2), we have

$$\begin{aligned}\left|\int_{\varepsilon<|y|<1} K(y)[f(x-y)-f(x)]\,dy\right| &\le \int_{\varepsilon<|y|<1} \frac{B}{|y|^n}\|\nabla f\|_{L_\infty}|y|\,dy \\ &\le B\|f\|_{L_\infty}\int_{\varepsilon<|y|<1}\frac{1}{|y|^{n-1}}\,dy.\end{aligned}$$

A change of variable into polar coordinates shows that

$$\int_{\varepsilon<|y|<1}\frac{1}{|y|^{n-1}}\,dy = c_n\int_\varepsilon^1 \frac{\rho^{n-1}}{\rho^{n-1}}\,d\rho = c_n(1-\varepsilon).$$

Hence for $0<\varepsilon<1$,

$$T_\varepsilon f(x) = \int_{|y|>\varepsilon} K(y)f(x-y)\,dy$$

is well-defined. Moreover, one can show that for $0<\eta<\varepsilon$, we have

$$|T_\varepsilon f(x) - T_\eta f(x)| \le C\|f\|_{L_\infty}(\varepsilon-\eta).$$

Hence $\lim_{\varepsilon\to 0+} T_\varepsilon f(x)$ exists, which completes the proof. $\square$

We call such an operator as the *Calderòn-Zygmund operator* (of convolution type). Here is the main theorem of this note, which was due to Calderón-Zygmund [1], which extends the result of M. Riesz[1] (1927)

Theorem 1.3. *Let K be a Calderón-Zygmund kernel and T be a Calderòn-Zygmund operator associated to the kernel K. If $1<p<\infty<$ there exists a constant $C=C(n,p)>0$ such that*

$$\|Tf\|_{L_p} \le C\|f\|_{L_p},$$

for all $f\in C_c^\infty$. Hence the operator T extends uniquely to the linear operator from L_p to itself.

Assuming Theorem 1.3, we prove (1.1). Recall the definition of K_{ij}:

$$K_{ij}(x) = \frac{nx_ix_j - \delta_{ij}|x|^2}{|x|^{n+2}}.$$

We check that K_{ij} is a Calderón-Zygmund kernel.

- (Size condition): Clearly $|K_{ij}(x)| \le (n+\delta)|x|^{-n}$;
- (Regularity condition): Observe that $|\nabla K_{ij}(x)| \le C|x|^{-n-1}$;

[1]There are Frigyes Riesz and Marcel Riesz. F. Riesz proved Riesz representation theorem and rising sun lemma. M. Riesz proved Riesz complex interpolation theorem.

- (Cancellation) Since K_{ij} is a homogeneous kernel of order n, it suffices to check that

$$\int_{\mathbb{S}^{n-1}} K_{ij}\, d\sigma = 0. \tag{1.3}$$

If $i \neq j$, note that

$$K_{ij}(x_1, \dots, -x_i, \dots, x_j, \dots, x_n) = -K_{ij}(x_1, \dots, x_n).$$

This implies that

$$\int_{\mathbb{S}^{n-1}} K_{ij}\, d\sigma = 0.$$

If $i = j$, a change of variable gives

$$\int_{\mathbb{S}^{n-1}} K_{ii}\, d\sigma = \int_{\mathbb{S}^{n-1}} K_{11}\, d\sigma.$$

Hence

$$\sum_{i=1}^{n} \int_{\mathbb{S}^{n-1}} K_{ii}\, d\sigma = \int_{\mathbb{S}^{n-1}} (|x|^2 - |x|^2)\, d\sigma(x) = 0$$

implies that K_{ij} has mean zero for all i, j.

Therefore, it follows from the Calderòn-Zygmund theorem (Theorem 1.3) that the operator $\mathcal{K}_{ij}$ defined on C_c^∞ extends to a bounded linear operator L_p to itself.

Let $u \in C_c^\infty$. Note that $\mathcal{N}[-\Delta u] = u$ (see Section A). Since

$$D_{ij}u = D_{ij}\mathcal{N}[-\Delta u] = \mathcal{K}_{ij}(-\Delta u) + \frac{\delta_{ij}}{n}\Delta u$$

for all $u \in C_c^\infty$, it follows that

$$\|D_{ij}u\|_{L_p} \lesssim \|\mathcal{K}_{ij}(-\Delta u)\|_{L_p} + \|\Delta u\|_{L_p} \lesssim \|\Delta u\|_{L_p}$$

for all $u \in C_c^\infty$, which proves (1.1).

The goal of this note is to show the proof of Theorem 1.3. The proof consists of five steps.

(1) We first prove that T is bounded on L_2 (L_2 has a more fruitful structure than L_p, e.g. Fourier transform, Hilbert spaces, ...)
(2) We prove that T is of weak type $(1, 1)$, that is, there exists a constant $C > 0$ such that

$$|\{x \in \mathbb{R}^n : |Tf(x)| > \lambda\}| \leq \frac{C}{\lambda}$$

for all $f \in C_c^\infty$.
(3) Using the Marcinkiewicz interpolation theorem, T is bounded on L_p for $1 < p < 2$.
(4) By duality, T is bounded on L_p for $2 < p < \infty$
(5) By density, we may extend T from C_c^∞ to L_p.

Step 2 is the highly nontrivial part of the proof. It uses the celebrated Calderón-Zygmund decomposition of a function $f \in L_1$, which we will see in detail in Section 2. Roughly speaking, the Calderón-Zygmund decomposition splits the function f into two pieces, the *good part* g and *bad part* b with respect to a fixed level α.

$$f = g + b.$$

More precisely, given $f \in L_1(\mathbb{R}^n)$ and $\alpha > 0$, there exist a collection of cubes $\{Q_j\}$ whose interiors are mutually disjoint, g, and b such that

$$f = g + b,$$

$$\left|\bigcup_j Q_j\right| \leq \frac{C}{\alpha}\|f\|_{L_1(\mathbb{R}^n)}$$

for some constant $C > 0$, $|g(x)| \leq c\alpha$, and

$$b(x) = 0 \quad \left(\bigcup_{j=1}^{\infty} Q_j\right)^c \quad \text{and} \quad \int_{Q_j} b(x)\,dx = 0 \quad \text{for each } j.$$

Why do we call g a good part? Maybe there are two reasons. First, $g \in L_1 \cap L_\infty$. In particular, for $1 < p < \infty$, we have

$$\|g\|_{L_p} \leq \|g\|_{L_1}^{1/p}\|g\|_{L_\infty}^{1-1/p} \leq \|f\|_{L_1}^{1/p}(2^n\alpha)^{1-1/p},$$

see Theorem 2.1. Second, if we know that T is bounded on L_2, then we can easily control the level set of Tg. Indeed, by Chebyshev's inequality and boundedness of T on L_2, we have

$$|\{x : |Tg(x)| > \alpha\}| \leq \frac{\|Tg\|_{L_2}^2}{\alpha^2} \leq \frac{C}{\alpha}\|f\|_{L_1}.$$

However, it is a bit hard to control the level set of Tb. To accomplish this goal, we need to use a 'remaining good structure' on b and the regularity conditions on the kernel K. Then the L_p-boundness of the operator follows from the Marcinkiewicz interpolation theorem and a standard duality argument.

2. Calderón-Zygmund decomposition

We say that a cube of the form

$$[2^k m_1, 2^k(m_1+1)) \times \cdots \times [2^k m_n, 2^k(m_n+1)),$$

where k, m_1, ..., $m_n \in \mathbb{Z}$ is *dyadic*.

Theorem 2.1. *Let $f \in L_1(\mathbb{R}^n)$ and $\alpha > 0$. There exist functions g, b, and collection of dyadic cubes $\{Q_j\}$ which are mutually disjoint satisfying the following properties:*

(a) $f = g + b$;
(b) $\|g\|_{L_1} \leq \|f\|_{L_1}$ *and* $\|g\|_{L_\infty} \leq 2^n\alpha$;
(c) $b = \sum_j b_j$, *where each* b_j *is supported in a dyadic cube* Q_j. *Furthermore, the cubes* Q_k *and* Q_j *are disjoint when* $j \neq k$;
(d) $\int_{Q_j} b_j(x)\,dx = 0$;
(e) $\|b_j\|_{L_1} \leq 2^{n+1}\alpha|Q_j|$;
(f) $\left|\bigcup_{j=1}^{\infty} Q_j\right| \leq \alpha^{-1}\|f\|_{L_1}$.

Proof. We decompose $\mathbb{R}^n$ into a mesh of disjoint dyadic cubes of the same size satisfying

$$|Q| \geq \frac{1}{\alpha}\|f\|_{L_1}.$$

Let $\mathcal{S}_0$ be the collection of these dyadic cubes. For each cube in $\mathcal{G}_0$, disect each side to get 2^n congruent cubes. Let $\mathcal{G}_1$ be the collection of such cubes. Choose a cube Q in $\mathcal{G}_1$ satisfying

$$\frac{1}{|Q|}\int_Q |f|\,dx > \alpha \tag{2.1}$$

and we collect these cubes, and we denote this collection by $\mathcal{S}_1$. Now we disect each element in $\mathcal{G}_1 \setminus \mathcal{S}_1$ into 2^n congruent cubes. Let $\mathcal{S}_2$ be the collection of such cubes and choose a cube Q in $\mathcal{G}_2$ satisfying (2.1). Continue this process and write

$$\mathcal{G} = \bigcup_{j=0}^{\infty} \mathcal{G}_j \quad \mathcal{Q} = \bigcup_j \mathcal{S}_j.$$

By construction, $\mathcal{Q}$ is countable. Of course, the collection could be empty. In this case, $g = f$ and $b = 0$.

We check several properties of $\{Q_j\}$. By the selection process, $\{Q_j\}$ is a collection of dyadic cubes which are disjoint. By (2.1), we have

$$|Q| < \frac{1}{\alpha}\int_Q |f|\,dx$$

for all $Q \in \mathcal{Q}$. Since $\{Q_j\}$ is disjoint, it follows that

$$\left|\bigcup_{j=1}^{\infty} Q_j\right| \le \sum_{j=1}^{\infty} |Q_j| < \frac{1}{\alpha}\sum_j \int_{Q_j} |f|\,dx \le \frac{1}{\alpha}\|f\|_{L_1}.$$

Define

$$b_j = \left(f - \frac{1}{|Q_j|}\int_{Q_j} f\,dx\right)\chi_{Q_j}$$

and

$$b = \sum_j b_j, \quad g = f - b.$$

For each $Q_j \in \mathcal{Q}$, there exists a unique parent $\hat{Q}_j$ in $\mathcal{G} \setminus \mathcal{Q}$ such that $Q_j \subset \hat{Q}_j$ $|\hat{Q}_j| = 2^n|Q_j|$. Since $\hat{Q}_j$ is not selected, we have

$$\frac{1}{|\hat{Q}_j|}\int_{\hat{Q}_j} |f|\,dx \le \alpha, \quad \text{i.e.,} \quad \frac{1}{|Q_j|}\int_{\hat{Q}_j} |f|\,dx \le 2^n\alpha.$$

So

$$\frac{1}{|Q_j|}\int_{Q_j} |f|\,dx \le \frac{1}{|Q_j|}\int_{\hat{Q}_j} |f|\,dx \le 2^n\alpha. \tag{2.2}$$

By definition of b_j, we have

$$\int_{Q_j} |b_j|\,dx \le \int_{Q_j} |f|\,dx + |Q_j|\left|\frac{1}{|Q_j|}\int_{Q_j} f\,dx\right| \le 2\int_{Q_j} |f|\,dx \le 2^{n+1}\alpha|Q_j|.$$

To get an information on good function g, note that

$$g = \begin{cases} f & \text{on } \mathbb{R}^n \setminus \bigcup_j Q_j, \\ \frac{1}{|Q_j|}\int_{Q_j} f\,dx & \text{on } Q_j. \end{cases}$$

By (2.2), we have

$$|g(x)| \le 2^n\alpha \quad \text{on } Q_j.$$

For $x \in \mathbb{R}^n \setminus \bigcup_j Q_j$, there exists a sequence $\{Q^{(k)}\}$ such that $x \in Q^{(k)}$ and $Q^{(k)} \in \mathcal{G}_k \setminus \mathcal{S}_k$ for all k. Then $\{Q^{(k)}\}$ shrinks nicely to x as $k \to \infty$. Hence it follows from the Lebesgue differentiation theorem that

$$\lim_{k\to\infty} \frac{1}{|Q^{(k)}|} \int_{Q^{(k)}} f\,dx = f(x)$$

a.e. on $\mathbb{R}^n \setminus \bigcup_j Q_j$. Since

$$\left| \frac{1}{|Q^{(k)}|} \int_{Q^{(k)}} f\,dx \right| \leq \int_{Q^{(k)}} |f|\,dx \leq \alpha,$$

it follows that $|f(x)| \leq \alpha$ a.e. on $\mathbb{R}^n \setminus \bigcup_j Q_j$. This implies that $|g(x)| \leq \alpha$ a.e. on $\mathbb{R}^n$. Finally, note that

$$\begin{aligned}
\int_{\mathbb{R}^n} |g(x)|\,dx &= \int_{\mathbb{R}^n \setminus \bigcup_j Q_j} |f(x)|\,dx + \int_{\bigcup_j Q_j} |g(x)|\,dx \\
&= \int_{\mathbb{R}^n \setminus \bigcup_j Q_j} |f(x)|\,dx + \sum_j \int_{Q_j} |g(x)|\,dx \\
&\leq \int_{\mathbb{R}^n \setminus \bigcup_j Q_j} |f(x)|\,dx + \sum_j \int_{Q_j} \left| \frac{1}{|Q_j|} \int_{Q_j} f\,dx \right| dz \\
&\leq \int_{\mathbb{R}^n \setminus \bigcup_j Q_j} |f(x)|\,dx + \sum_j \int_{Q_j} |f|\,dx \\
&= \|f\|_{L_1}.
\end{aligned}$$

This completes the proof of Theorem 2.1. □

Remark. Calderón-Zygmund decomposition can be found in many places in PDEs paper. Good examples are 'reverse Hölder estimate (Gehring lemma)' and 'a crawling ink spot lemma'. We will see these examples in the next week.

3. Proof of Theorem 1.3

In this section, we prove Theorem 1.3. For simplicity, we first assume that T is bounded on L_2 (see Proposition 3.2). This will be proved at the end of this section.

Proposition 3.1. *The operator T is of weak type $(1,1)$.*

Proof. We will show that there exists $C = C(n) > 0$ such that

$$|\{x : |Tf(x)| > \lambda\}| \leq \frac{C}{\lambda} \|f\|_{L_1(\mathbb{R}^n)}$$

for all $\lambda > 0$.

Apply Calderón-Zygmund decomposition to f with the level λ. Then there exist a collection of cubes $\mathcal{Q} = \{Q_j\}$ and decomposition $f = g + b$ given in Theorem 2.1. Note that

$$\{x : |Tf(x)| > \lambda\} \subset \{x : |Tg(x)| > \lambda/2\} \cup \{x : |Tb(x)| > \lambda/2\}.$$

For the estimate of the good part, it follows from the L_2-boundedness of T that

$$\begin{aligned}
|\{x : |Tg(x)| > \lambda/2\}| &\leq \frac{4}{\lambda^2} \|Tg\|_{L_2}^2 \\
&\leq \frac{C}{\lambda^2} \|g\|_{L_2}^2
\end{aligned}$$

$$\leq \frac{C}{\lambda^2}\|g\|_{L_1}\|g\|_{L_\infty}$$
$$\leq \frac{C}{\lambda}\|g\|_{L_1}$$

since $|g| \leq \lambda$ a.e. To estimate the bad part, we will crucially use mean-zero property of b_j on Q_j and the Hörmander condition of K. For each $Q \in \mathcal{Q}$, let Q^* be the dilate of Q by $2\sqrt{n}$.

By Chebyshev's inequality and the property of the collection $\mathcal{Q}$, we have

$$\begin{aligned}
|\{x : |Tb(x)| > \lambda/2\}| &\leq \left|\bigcup_{Q\in\mathcal{Q}} Q^*\right| + \left|\left\{x \in \mathbb{R}^n \setminus \bigcup_{Q\in\mathcal{Q}} Q^* : |Tb(x)| > \lambda/2\right\}\right| \\
&\leq C\sum_{Q\in\mathcal{Q}}|Q| + \frac{2}{\lambda}\int_{\mathbb{R}^n\setminus\bigcup Q^*}|(Tb)(x)|\,dx \\
&\leq \frac{C}{\lambda}\|f\|_{L_1} + \frac{2}{\lambda}\int_{\mathbb{R}^n\setminus\bigcup Q^*}|(Tb)(x)|\,dx.
\end{aligned}$$

Since $b = \sum_j b_j$ and $\mathbb{R}^n \setminus \bigcup_j Q_j^* \subset \mathbb{R}^n \setminus Q_k^*$ for all k, it follows that

$$\begin{aligned}
\frac{2}{\lambda}\int_{\mathbb{R}^n\setminus\bigcup_j Q_j^*}|(Tb)(x)|\,dx &\leq \frac{2}{\lambda}\int_{\mathbb{R}^n\setminus\bigcup_j Q_j^*}\sum_j|(Tb_j)(x)|\,dx \\
&\leq \frac{2}{\lambda}\sum_j\int_{\mathbb{R}^n\setminus Q_j^*}|(Tb_j)(x)|\,dx.
\end{aligned}$$

Let y_j be the center of the cube Q_j. Since $\int_{Q_j} b_j\,dx = 0$, it follows that for any $x \in \mathbb{R}^n \setminus Q_j^*$, we have

$$\begin{aligned}
(Tb_j)(x) &= \int_{Q_j} K(x-y)b_j(y)\,dy \\
&= \int_{Q_j}(K(x-y) - K(x-y_j))b_j(y)\,dy.
\end{aligned}$$

Hence by Fubini's theorem, we have

$$\begin{aligned}
\int_{\mathbb{R}^n\setminus Q_j^*}|(Tb_j)(x)|\,dx &\leq \int_{\mathbb{R}^n\setminus Q_j^*}\int_{Q_j}|K(x-y) - K(x-y_j)||b_j(y)|\,dy\,dx \\
&= \int_{Q_j}|b_j(y)|\left(\int_{\mathbb{R}^n\setminus Q_j^*}|K(x-y) - K(x-y_j)|\,dx\right)dy.
\end{aligned}$$

Note that for $x \in (Q_j^*)^c$ and $y \in Q_j$, we have

$$|y - y_j| \leq \frac{\ell(Q_j)}{2}\sqrt{n} \quad \text{and} \quad |x - y_j| \geq \sqrt{n}\ell(Q_j),$$

where $\ell(Q_j)$ is the side length of the cube Q_j. Hence

$$|x - y_j| \geq 2|y - y_j|,$$

(see Figure 3.1). This implies that

$$\int_{\mathbb{R}^n\setminus Q_j^*}|K(x-y) - K(x-y_j)|\,dx \leq \int_{|x-y_j|\geq 2|y-y_j|}|K(x-y) - K(x-y_j)|\,dx.$$

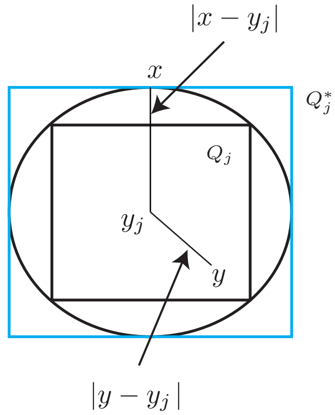

FIGURE 3.1. Applying Hörmander condition

Then by the Hörmander condition (if we take $x-y_j$ instead of x and $y-y_j$ instead of y), we get

$$\int_{|x-y_j|\geq 2|y-y_j|} |K(x-y)-K(x-y_j)|\,dx \leq C.$$

Hence

$$\int_{\mathbb{R}^n\setminus Q_j^*} |(Tb_j)(x)|\,dx \leq C\int_{Q_j} |b_j(y)|\,dy.$$

Therefore, we conclude that

$$|\{x : |Tb(x)| > \lambda/2\}| \leq \frac{C}{\lambda}\|f\|_{L_1}.$$

So

$$|\{x : Tf(x)| > \lambda\}| \leq \frac{C}{\lambda}\|f\|_{L_1},$$

which proves that T is of weak type $(1,1)$. ☐

Since

$$\|Tf\|_{L_2} \leq C\|f\|_{L_2}$$

and

$$|\{x : |Tf(x)| > \lambda\}| \leq \frac{C}{\lambda}\|f\|_{L_1}$$

for all $f \in C_c^\infty$, it follows from the Marcinkiewicz interpolation theorem that for $1 < p < 2$, there exists a constant $C = C(n,p) > 0$ such that

$$\|Tf\|_{L_p} \leq C\|f\|_{L_p}$$

for all $f \in C_c^\infty$.

Next, suppose that $2 < p < \infty$. For $g \in C_c^\infty$, define

$$T^*g(x) = \lim_{\varepsilon\to 0+}\int_{|x-y|>\varepsilon} K^*(x-y)g(y)\,dy,$$

where $K^*(x) = \overline{K(-x)}$. Note that

$$\int_{\mathbb{R}^n} (Tf)g\,dx = \int_{\mathbb{R}^n} f(T^*g)\,dx$$

for all $f, g \in C_c^\infty$. Since K^* satisfy Calderón-Zygmund condition, it follows that $\|T^*g\|_{L_{p'}} \le C\|g\|_{L_{p'}}$ for some $C = C(n,p) > 0$. Hence

$$\left|\int_{\mathbb{R}^n} (Tf)g\,dx\right| \le \|f\|_{L_p}\|T^*g\|_{L_{p'}} \le C\|f\|_{L_p}\|g\|_{L_{p'}}$$

for all $f, g \in C_c^\infty$. Hence by duality in L_p, we conclude that

$$\|Tf\|_{L_p} \le C\|f\|_{L_p} \quad \text{for all } f \in C_c^\infty$$

for some constant $C = C(n,p) > 0$.

It remains to show

Proposition 3.2. *There exists a constant $C > 0$ such that*

$$\|Tf\|_{L_2} \le C\|f\|_{L_2}$$

for all $f \in C_c^\infty$.

Proof. Fix $0 < \varepsilon < N < \infty$ and introduce the following truncated operator

$$(T_{\varepsilon,N}f)(x) = \int_{\mathbb{R}^n} K(y)\chi_{\varepsilon<|y|<N}(y)f(x-y)\,dy,$$

which is well-defined for $f \in C_c^\infty$. For $\xi \in \mathbb{R}^n$, define

$$m_{\varepsilon,N}(\xi) = \frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n} e^{-ix\cdot\xi}\chi_{\varepsilon<|x|<N}K(x)\,dx.$$

We will show that there exists $C > 0$ independent of ε and N such that

$$\sup_{\varepsilon,N}\|m_{\varepsilon,N}\|_{L_\infty} \le C. \tag{3.1}$$

If we prove this, then it follows from the Plancherel theorem that

$$\|T_{\varepsilon,N}f\|_{L_2} = \|m_{\varepsilon,N}\hat{f}\|_{L_2} \le C\|\hat{f}\|_{L_2} = \|f\|_{L_2}$$

for all $f \in C_c^\infty$. Here the constant C is independent of ε and N. Moreover, it follows from Proposition 1.2 that

$$\lim_{\varepsilon\to0,N\to\infty} T_{\varepsilon,N}f(x) = f(x)$$

for all $f \in C_c^\infty$. Hence by Fatou's lemma, we conclude that

$$\|Tf\|_{L_2} \le C\|f\|_{L_2}$$

for all $f \in C_c^\infty$.

Hence it remains to show that (3.1) holds. We split the integral in the definition of $m_{\varepsilon,N}$ into two regions $|x| < 3\pi|\xi|^{-1}$ and $|x| \ge 3\pi|\xi|^{-1}$. The region $\{\varepsilon < |x| < 3\pi|\xi|^{-1}\}$ is relatively easy. Indeed, by using the cancellation condition and

$$|e^{-ix\cdot\xi} - 1| = \sqrt{2(1-\cos x)} = |2\sin(x/2)| \le |x|,$$

we have

$$\left|\int_{\varepsilon<|x|<3\pi|\xi|^{-1}} e^{-ix\cdot\xi}K(x)\,dx\right| = \left|\int_{\varepsilon<|x|<3\pi|\xi|^{-1}} (e^{-ix\cdot\xi}-1)K(x)\,dx\right|$$

$$\leq C \int_{|x|<3\pi|\xi|^{-1}} \sin((x\cdot\xi)/2)||K(x)|\,dx$$
$$\leq C|\xi| \int_{|x|<3\pi|\xi|^{-1}} \frac{1}{|x|^{n-1}}\,dx = C.$$

To estimate the second part, we first observe that

$$-1 = e^{-\pi i} = e^{-\pi i \frac{\xi}{|\xi|^2}\cdot\xi}.$$

A change of variable shows that

$$I(\xi) = \int_{3\pi|\xi|^{-1}<|x|<N} K(x)e^{-ix\cdot\xi}\,dx = -\int_{3\pi|\xi|^{-1}<|x|<N} K(x)e^{-i(x+\pi(\xi/|\xi|^2))\cdot\xi}\,dx$$
$$= -\int_{3\pi|\xi|^{-1}<|x-\pi\xi/|\xi|^2|<N} K\left(x-\frac{\pi\xi}{|\xi|^2}\right)e^{-ix\cdot\xi}\,dx.$$

Write $z = \frac{\pi\xi}{|\xi|^2}$ for convenience. Then

$$I(\xi) = \frac{1}{2}\int_{3\pi|\xi|^{-1}<|x|<N} K(x)e^{-ix\cdot\xi}\,dx - \frac{1}{2}\int_{3\pi|\xi|^{-1}<|x-z|<N} K\left(x-z\right)e^{-ix\cdot\xi}\,dx.$$

We carefully estimate this quantity into five terms.

$$I(\xi) = J_1(\xi)+J_2(\xi)+J_3(\xi)+J_4(\xi)+J_5(\xi),$$

where

$$J_1(\xi) = \frac{1}{2}\int_{3\pi|\xi|^{-1}<|x-z|<N} (K(x)-K(x-z))e^{-ix\cdot\xi}\,dx$$
$$J_2(\xi) = \frac{1}{2}\int_{\substack{3\pi|\xi|^{-1}<|x|<N\\ |x-z|\leq 3\pi|\xi|^{-1}}} K(x)e^{-ix\cdot\xi}\,dx$$
$$J_3(\xi) = \frac{1}{2}\int_{\substack{3\pi|\xi|^{-1}<|x|<N\\ |x-z|\geq N}} K(x)e^{-ix\cdot\xi}\,dx$$
$$J_4(\xi) = \frac{1}{2}\int_{\substack{3\pi|\xi|^{-1}<|x-z|<N\\ |x|\leq 3\pi|\xi|^{-1}}} K(x)e^{-ix\cdot\xi}\,dx$$
$$J_5(\xi) = \frac{1}{2}\int_{\substack{3\pi|\xi|^{-1}<|x-z|<N\\ |x|\geq N}} K(x)e^{-ix\cdot\xi}\,dx.$$

To estimate $J_1(\xi)$, note that $|z| = \pi|\xi|^{-1}$. Since $3|z| \leq |x-z|$, it follows that $|x| \geq 2|z|$. Hence

$$|J_1(\xi)| \lesssim 1$$

by the Hörmander condition. To estimate $J_2(\xi)$, note that $3|z| \leq |x|$ and $|x-z| \leq 3|z|$ imply $3|z| \leq |x| \leq 4|z|$. Hence

$$|J_2(\xi)| \lesssim \int_{3|z|}^{4|z|} \frac{1}{r}\,dr \approx 1.$$

Similarly, we get

$$|J_4(\xi)| \lesssim 1.$$

To estimate J_3, we note that

$$3|z| \leq |x| \leq N \quad \text{and} \quad |x-z| \geq N$$

implies

$$\frac{2}{3}N \leq N - |z| \leq |x| \leq N.$$

Then

$$|J_3(\xi)| \lesssim \int_{\frac{2}{3}N}^{N} \frac{1}{r}\,dr \approx 1.$$

Similarly,

$$|J_5(\xi)| \lesssim \int_{N}^{\frac{4}{3}N} \frac{1}{r}\,dr \approx 1.$$

These estimates imply that

$$|I(\xi)| \lesssim 1$$

uniformly in ξ. This completes the proof of Proposition 3.2. □

4. Mean oscillation estimates

In this section, we give another approach to obtain (1.1) without using singular integral theory. This approach was proposed by Krylov [6]. For convenience, we introduce some notation. If $A \subset \mathbb{R}^n$ with $0 < |A| < \infty$, we define

$$\fint_A f\,dx = \frac{1}{|A|}\int_A f\,dx$$

and

$$(f)_A = \fint_A f\,dx.$$

Suppose that $n \geq 3$ and set $f = -\Delta u$, where $u \in C_c^\infty$. Recall that

$$D_{ij}u(x) = C_1(n)\mathcal{K}_{ij}f(x) - \frac{\delta_{ij}}{n}f(x),$$

where

$$\lim_{r\to 0+}\mathcal{K}_{ij}f(x) = \lim_{r\to 0+}\int_{|y|\geq r} K_{ij}(y)f(x+y)\,dy$$

and

$$K_{ij}(x) = \frac{nx_ix_j - \delta_{ij}|x|^2}{|x|^{n+2}}.$$

Let $r > 0$ and choose a cut-off function $\zeta \in C_c^\infty$ such that $\zeta = 1$ in B_{3r} and $\zeta = 0$ outside B_{4r}. Decompose u and f into

$$g = \zeta f \quad \text{and} \quad h = (1-\zeta)f$$

and

$$v = \mathcal{N}g \quad \text{and} \quad w = \mathcal{N}h.$$

It is easy to see that v and w are infinitely differentiable,

$$-\Delta v = g \quad \text{and} \quad -\Delta w = h,$$

(see Section A). Note that g is localized and h removes a singularity near 0 in the integral representation of w. We first estimate the mean oscillation of $D_{ij}v$. Integration by part shows that

$$\int_{\mathbb{R}^n} |D_{ij}v|^2\,dx \le C\int_{\mathbb{R}^n} |g|^2\,dx \le C\int_{B_{4r}} |f|^2\,dx.$$

Note that

$$⨍_B |u-u_B|^2\,dx = ⨍_B |u|^2\,dx - (u_B)^2 \le ⨍_B |u|^2\,dx$$

for any set B in $\mathbb{R}^n$ with $0<|B|<\infty$. Moreover, for any $x_0 \in B_r$, Jensen's inequality and the above estimate show that

$$\begin{aligned} ⨍_{B_r} |D_{ij}v-(D_{ij}v)_{B_r}|\,dx &\le \left(⨍_{B_r} |D_{ij}v-(D_{ij}v)_{B_r}|^2\,dx\right)^{1/2} \\ &\le \left(⨍_{B_r} |D_{ij}v|^2\,dx\right)^{1/2} \\ &\le C\left(⨍_{B_{4r}} |f|^2\,dx\right) \le CM^{1/2}(|f|^2)(x_0), \end{aligned}$$

where M is the Hardy-Littlewood maximal function. To estimate the mean oscillation of $D_{ij}w$, note first that since $h=0$ in B_{3r}, it follows that for $x\in B_r$, we have

$$\begin{aligned} w_{x_ix_j}(x) &= \int_{|y|\ge 2r} K_{ij}(y)h(x+y)\,dy, \\ w_{x_ix_jx_k}(x) &= \int_{|y|\ge 2r} K_{ij}(z)h_{x_k}(x+y)\,dy = -\int_{|y|\ge 2r} K_{ijx^k}(y)h(x+y)\,dy. \end{aligned}$$

Note that $|x|<r$ and $|y|\ge 2r$ imply $2|y|\ge |x+y|\ge r$. Since $|\nabla K_{ij}(y)|\le C|y|^{-n-1}$, it follows that

$$\begin{aligned} |w_{x_ix_jx_k}(x)| &\le C\int_{|y|\ge 2r} |x+y|^{-n-1}|h(x+y)|\,dy \\ &\le C\int_{|x+y|\ge r} |x+y|^{-n-1}|h(x+y)|\,dy \\ &= C\int_{|z|\ge r} |z|^{-n-1}|h(z)|\,dz. \end{aligned}$$

Set

$$\psi(\rho) = \int_{\mathbb{S}^{n-1}} |h(\rho\omega)|\,d\sigma(\omega).$$

Observe that

$$\frac{\partial}{\partial\rho}\int_{B_\rho} |h(y)|\,dy = \frac{\partial}{\partial\rho}\int_0^\rho r^{n-1}\psi(r)\,dr = \rho^{n-1}\psi(\rho).$$

Then a change of variable into polar coordinate and integration by part give

$$\begin{aligned} \int_{|z|\ge r} |z|^{-n-1}|h(z)|\,dz &= C\int_r^\infty \rho^{-n-1}\rho^{n-1}\left(\int_{\mathbb{S}^{n-1}} |h(\rho\omega)|\,d\sigma(\omega)\right)d\rho \\ &= C\int_r^\infty \rho^{-n-1}\left(\frac{\partial}{\partial\rho}\int_{B_\rho} |h(y)|\,dy\right)d\rho \end{aligned}$$

$$
\begin{aligned}
&= C\int_r^\infty \rho^{-n-2}\int_{B_\rho}|h(y)|\,dyd\rho - Cr^{-n-1}\int_{B_r}|h(y)|\,dy\\
&\le C\int_r^\infty \rho^{-2}⨍_{B_\rho}|h(y)|\,dyd\rho\\
&\le CMh(x_0)\int_r^\infty \rho^{-2}\\
&\le Cr^{-1}Mf(x_0) \le Cr^{-1}M^{1/2}(|f|^2)(x_0).
\end{aligned}
$$

Hence by the mean value theorem, we get

$$⨍_{B_r}|D_{ij}w - (D_{ij}w)_{B_r}|\,dx \le Nr\sup_{B_r}|\nabla(D_{ij}w)| \le CM^{1/2}(|f|^2)(x_0).$$

Therefore, we get the following estimate

$$⨍_{B_r}|D_{ij}u - (D_{ij}u)_{B_r}|\,dx \le CM^{1/2}(|f|^2)(x_0),$$

where the constant C depends only on n. Hence for any x_0 and B containing x_0, we have

$$⨍_{B}|D_{ij}u - (D_{ij}u)_{B_r}|\,dx \le CM^{1/2}(|f|^2)(x_0). \tag{4.1}$$

Now we introduce the definition to handle the left-hand side.

Definition 4.1. For a locally integrable function $f : \mathbb{R}^n \to \mathbb{C}$, we define

$$f^{\#}(x) = \sup_{B;x\in B}⨍_B |f - (f)_B|\,dy,$$

where the supremum is taken over all ball B containing x.

The following theorem was shown by Fefferman-Stein [2] (see [10, Theorem 2, Chapter IV]). The proof also involves a delicate Calderòn-Zygmund decomposition.

Theorem 4.2. *Let $1 < p < \infty$. Then there exists a constant $C = C(n, p) > 0$ such that*

$$\|f\|_{L_p} \le C\|f^{\#}\|_{L_p}$$

for all $f \in L_p$.

Now using the Fefferman-Stein theorem, we prove estimate (1.1). By (4.1), we have

$$(D_{ij}u)^{\#}(x) \le CM^{1/2}(|f|^2)(x).$$

Hence by the Fefferman-Stein theorem and Hardy-Littlewood maximal function theorem, for $2 < p < \infty$, we have

$$
\begin{aligned}
\|D_{ij}u\|_{L_p} &\le C\|(D_{ij}u)^{\#}\|_{L_p}\\
&\le C\|M^{1/2}(|f|^2)\|_{L_p}\\
&= C\||f|^2\|_{L_{p/2}}^{1/2}\\
&\le C\||f|^2\|_{L_{p/2}}^{1/2} = C\|f\|_{L_p}
\end{aligned}
$$

for some constant $C = C(n,p) > 0$. This proves the estimate (1.1) when $2 < p < \infty$. To prove that the estimate holds for $1 < p < 2$, we use a duality argument. Let $u, v \in C_c^\infty$. Then by Hölder's inequality and the previous estimate, we have

$$\begin{aligned}\int_{\mathbb{R}^n} u_{x_ix_j} \Delta v \, dx &= \int_{\mathbb{R}^n} (\Delta u) v_{x_ix_j} \, dx \\ &\leq \|\Delta u\|_{L_p} \|D_{ij} v\|_{L_{p'}} \\ &\leq C \|\Delta u\|_{L_p} \|\Delta v\|_{L_{p'}}.\end{aligned}$$

Then the desired result follows from the following density result and the duality of L_p.

Lemma 4.3. *If $1 < p < \infty$, then ΔC_c^∞ is dense in L_p.*

Proof. Let $\varepsilon > 0$ be given and let $f \in L_p$. There exists $\phi \in C_c^\infty$ such that $\|f - \phi\|_{L_p} < \varepsilon$. Let $K = \frac{1}{\omega_n} \chi_{B_1}$. For $\delta > 0$, define $K_\delta(x) = \delta^n K(\delta x)$. Note that

$$\|K_\delta * \phi\|_{L_p} \leq (\|K\|_{L_1} \|\phi\|_{L_1})^{1/p} (\delta^n \|\phi\|_{L_\infty})^{1-1/p}.$$

Letting $\delta \to 0$, we see that $K_\delta * \phi \to 0$ in L_p. Let Γ be the fundamental solution of $-\Delta$. If we define $\psi_\delta = \Gamma * (\phi * K_\delta - \phi)$, then note that $K_\delta * \Gamma(x) = \Gamma(x)$ if $|x| > 1/\delta$. So $\psi_\delta \in C_c^\infty$. Since $-\Delta \psi_\delta = \phi * K_\delta - \phi$, it follows that

$$\|\Delta \psi_\delta - f\|_{L_p} \leq \|\phi * K_\delta\|_{L_p} + \|\phi - f\|_{L_p} < 2\varepsilon$$

by choosing sufficiently small $\delta > 0$ so that $\|\phi * K_\delta\|_{L_p} < \varepsilon$. This completes the proof of Lemma 4.3. □

Note that we proved the Calderón-Zygmund estimate when $n \geq 3$. One can easily show that the Calderón-Zygmund estimate also holds when $n = 2$ by considering $u_k(x_1, x_2, x_3) = u(x_1, x_2)\zeta(x_3/k)$, where $\zeta \in C_c^\infty(\mathbb{R})$ such that $\zeta(0) = 1$.

Remark. There is another approach to show the Calderón-Zygmund estimate (1.1) due to Wang [11]. This approach uses a refined Vitali covering lemma and Hardy-Littlewood maximal function without using Calderón-Zygmund decomposition.

5. Further results

There are different formulations of Calderón-Zygmund operators. A good reference on this topic is Stein [10]. Here we introduce a formulation due to Krylov [5].

Definition 5.1. Let $k \in \mathbb{N}$ and $(\mathbb{Q}_m; m \in \mathbb{Z})$ be a sequence of partitions of $\mathbb{R}^n$, whose elements are bounded Borel subsets. We call $(\mathbb{Q}_m, m \in \mathbb{Z})$ a *filtration of partitions* if

(i) the partitions become finer as m increases:

$$\inf_{Q \in \mathbb{Q}_m} |Q| \to \infty \quad \text{as } m \to -\infty$$

and

$$\sup_{Q \in \mathbb{Q}_m} \operatorname{diam} |Q| \to 0 \quad \text{as } m \to \infty;$$

(ii) the partitions are nested: for each m and $Q \in \mathbb{Q}_m$, there is a (unique) set $Q' \in \mathbb{Q}_{m-1}$ such that $Q \subset Q'$;

(iii) the following regularity property holds: for Q and Q' as in (ii), we have $|Q'| \leq C_0 |Q|$, where C_0 is a constant independent of m, Q, and Q'.

Example 5.2. Let

$$\mathbb{Q}_m = \{[i_0 4^{-m}, (i_0+1)4^{-m}) \times Q_m(i_1, \ldots, i_n) : i_0, i_1, \ldots, i_n \in \mathbb{Z}\},$$

where

$$Q_m(i_1, \ldots, i_n) = [i_1 2^{-m}, (i_1+1)2^{-m}) \times \cdots \times [i_n 2^{-m}, (i_d+1)2^{-m}).$$

Then $(\mathbb{Q}_m)_{m\in\mathbb{Z}}$ is a filtration of partitions on $\mathbb{R}^{1+n}$.

This filtration of partitions is adapted to second-order parabolic equations.

Definition 5.3. Let $(Q_m, m \in \mathbb{Z})$ be a filtration of partitions, and for each $x, y \in \mathbb{R}^n$, $x \neq y$, let $K(x,y)$ be a bounded operator from F into G. We say that K is a *Calderón-Zygmund kernel* relative to $(\mathbb{Q}_m, m \in \mathbb{Z})$ if

(1) there is a number $1 < p_0 < \infty$ such that for any x and any $r > 0$, $K(x, \cdot) \in L_{p_0,\mathrm{loc}}(B_r^c(x), L(F,G))$;
(2) for every $y \in \mathbb{R}^n$, the function $|K(x,y) - K(x,z)|$ is measurable as a function of (x,z) on the set $\mathbb{R}^{2n} \cap \{(x,z) : x \neq z, x \neq y\}$;
(3) there is a constant $C_0 \geq 1$ and for each $Q \in \bigcup_{m\in\mathbb{Z}} \mathbb{Q}_m$, there is a closed set Q^* with the properties $Q' \subset Q^*$, $|Q^*| \leq C_0|Q|$, and

$$\int_{\mathbb{R}^n \setminus Q^*} |K(x,y) - K(x,z)|\, dx \leq C_0,$$

whenever $y, z \in Q$.

Now we present the main theorem of Krylov [5]. Using this theorem, he obtained mixed norm estimates of second-order parabolic equations whose leading coefficient is merely measurable in t.

Theorem 5.4. *Let $1 < p < \infty$ and $A : L_p(\mathbb{R}^n; F) \to L_p(\mathbb{R}^n; G)$ be a bounded linear operator. Assume that if $f \in C_c^\infty(\mathbb{R}^n; F)$, then for almost any x outside the support of f, we have*

$$Af(x) = \int_{\mathbb{R}^n} K(x,y) f(y)\, dy,$$

where $K(x,y)$ is a Calderón-Zygmund kernel relative to a filtration of partitions. Then the operator A is uniquely extendable to a bounded operator from $L_q(\mathbb{R}^n; F)$ to $L_q(\mathbb{R}^n; G)$ for any $1 < q \leq p$, and A is of weak type $(1,1)$ on smooth functions with compact support.

Appendix A. Newtonian potential

Let $n \geq 2$ and $h \in C_c^\infty(\mathbb{R}^n)$. Recall the fundamental solution of $-\Delta$:

$$\Phi(x) = \begin{cases} \frac{1}{n(n-2)\omega_n} \frac{1}{|x|^{n-2}} & \text{if } n \geq 3, \\ -\frac{1}{2\pi} \ln|x| & \text{if } n = 2, \end{cases}$$

Theorem A.1. *For $f \in C_c^\infty(\mathbb{R}^n)$, $n \geq 2$, the Newtonian potential*

$$\mathcal{N}[f](x) = \int_{\mathbb{R}^n} \Phi(y) f(x-y)\, dy$$

is C^∞ and $-\Delta(\mathcal{N}[f]) = f$. In particular, $\mathcal{N}[-\Delta u] = u$ for all $u \in C_c^\infty(\mathbb{R}^n)$.

Proof. Related to our note, we only prove $\mathcal{N}[-\Delta u] = u$ for $u \in C_c^\infty(\mathbb{R}^n)$ since proving another assertion is essentially the same. Also, for simplicity, we mainly focus on the case $n \geq 3$. Note that

$$\nabla\Phi(x) = -\frac{1}{n\omega_n}\frac{x}{|x|^n}.$$

For $\varepsilon > 0$, integration by part gives

$$\begin{aligned}\int_{\mathbb{R}^n\setminus B_\varepsilon} \Phi(y)\Delta_y u(x-y)\,dy = &-\int_{\mathbb{R}^n\setminus B_\varepsilon} \nabla\Phi(y)\cdot\nabla_y u(x-y)\,dy \\ &+ \int_{\partial B_\varepsilon} \Phi(y)\nabla_y u(x-y)\cdot\left(-\frac{y}{|y|}\right)\,d\sigma(y)\end{aligned}$$

Note that

$$\left|\int_{\partial B_\varepsilon} \Phi(y)\nabla_y u(x-y)\cdot\left(-\frac{y}{|y|}\right)\,d\sigma(y)\right| \leq c\frac{1}{\varepsilon^{n-2}}\varepsilon^{n-1}\|\nabla u\|_{L_\infty} = O(\varepsilon).$$

Taking integration by part again, we get

$$\begin{aligned}-\int_{\mathbb{R}^n\setminus B_\varepsilon} \nabla\Phi(y)\cdot\nabla_y u(x-y)\,dy = &-\int_{\partial B_\varepsilon} u(x-y)\nabla\Phi(y)\cdot\left(-\frac{y}{|y|}\right)\,d\sigma(y) \\ &+ \int_{\mathbb{R}^n\setminus B_\varepsilon} \Delta\Phi(y)u(x-y)\,dy.\end{aligned}$$

Since Φ is harmonic in $\mathbb{R}^n \setminus B_\varepsilon$, the second integral on the right-hand side is zero. Also,

$$\nabla\Phi(y)\cdot\left(-\frac{y}{|y|}\right) = \frac{1}{n\omega_n\varepsilon^{n-1}} \quad \text{on } \partial B_\varepsilon,$$

it follows that

$$\int_{\mathbb{R}^n\setminus B_\varepsilon} \Phi(y)\Delta_y u(x-y)\,dy = -\frac{1}{n\omega_n\varepsilon^{n-1}}\int_{\partial B_\varepsilon} u(x-y)\,d\sigma(y).$$

Since $u \in C_c^\infty$, letting $\varepsilon \to 0$, we conclude that

$$\mathcal{N}[\Delta u](x) = -u(x),$$

which completes the proof. □

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