# APPLICATION OF CALDERÓN-ZYGMUND DECOMPOSITION

HYUNWOO KWON

Abstract. We present several applications of Calderón-Zygmund decompositions related to PDEs.

## 1. Introduction

Let us consider the following elliptic equations in divergence form

$$-\operatorname{div}(A\nabla u) = 0 \quad \text{in } B_2 \tag{1.1}$$

and elliptic equations in non-divergence form

$$a^{ij}D_{ij}u = 0 \quad \text{in } B_2. \tag{1.2}$$

Here the leading matrix $A(x) = (a^{ij}(x))$ satisfies uniformly ellipticity: there exists $\delta \in (0,1)$ such that

$$\delta|\xi|^2 \leq a^{ij}(x)D_{ij}\xi_i\xi_j \leq \delta^{-1}|\xi|^2$$

for all $x \in B_2$ and $\xi \in \mathbb{R}^n$.

In 1900, Hilbert proposed twenty problems that led to the development of modern mathematics. One of the problems related to PDEs is the 19th problem, which states that if $F$ is smooth and $u$ satisfies

$$I[u] = \min_v I[v],$$

where

$$I[u] = \int_\Omega F(v)\,dx,$$

then $u$ is smooth. After the development of the Schauder theory, the problem is reduced to proving that if $A$ is merely measurable and $u \in W^{1,2}(B_2)$ is a weak solution of (1.1), then $u \in C^\alpha(B_1)$ for some $\alpha \in (0,1)$. This was proved by De Giorgi and Nash almost at the same time. The argument of De Giorgi [5] was simplified by Moser [20]. We also note that the parabolic version was first proved by Nash [21]. This theorem is now called the De Giorgi-Nash-Moser theorem.

The next natural question is to seek an analog result of the De Giorgi-Nash-Moser theorem to elliptic equations of non-divergence form (1.2). This was obtained by Krylov and Safonov [18], first proved by a probabilistic argument and later by PDE argument.

However, these theorems only work when the equation is real-valued scalar equation. Hence one could naturally ask whether we do have Hölder regularity of solutions to elliptic systems, but this turns out to be false by a famous counterexample due to De Giorgi [6] (see Nevertheless, we could prove another regularity theorem, the Gehring lemma.

Related to our purpose of this note, we mainly focus on proving machinery that could be used in PDEs. First, we prove the crawling ink spot lemma. We also give a brief scheme to apply this theorem to the Krylov-Safonov theorem. Second, we prove the John-Nirenberg theorem [16] on bounded mean oscillation spaces (BMO). We apply this theorem to a proof of the De Giorgi-Nash-Moser theorem. Finally, we state a Gehring lemma and we apply this theorem to elliptic systems. All of these things crucially use the idea of Calderón-Zygmund decomposition.

## 2. Crawling ink spot lemma

We first derive a crawling ink spot lemma (or growing lemma, covering lemma), which was first observed by Krylov and Safonov. The name was due to Landis who gave another proof of the De Giorgi-Nash-Moser theorem.

Here we follow an argument of Caffarelli-Peral [2].

**Theorem 2.1.** *Let $Q$ be a bounded cube in $\mathbb{R}^n$. Assume that there exist $\delta \in (0,1)$, measurable sets $A$ and $B$ satisfying $A \subset Q$ and $|A| < \delta|Q|$. Then there exists a sequence of disjoint dyadic cubes obtained from $Q$, $\{Q_k\}$ such that*

- $|A \setminus \bigcup_k Q_k| = 0$,
- $|A \cap Q_k| > \delta|Q_k|$, *and*
- $|A \cap \hat{Q}_k| < \delta|\hat{Q}_k|$ *if $Q_k$ is a dyadic subdivision of $\hat{Q}_k$.*

*Proof.* The first part is a direct application of Calderón-Zygmund decomposition to $\chi_A$ with height $\delta$. Since the idea of Calderón-Zygmund decomposition is important, we prove an adapted version to this theorem.

First, we divide $Q$ into $2^N$ $(Q_1^j)$ dyadic cubes and let $\mathcal{G}_1$ be the collection of such cubes. Choose those for which

$$|Q_1^j \cap A| > \delta|Q_1^j|$$

and let $\mathcal{S}_1$ be the collection of such cubes. For each cube in $\mathcal{G}_1 \setminus \mathcal{S}_1$, disect each side of the cube into $2^n$ dyadic cubes. Choose $Q_2^j$ satisfying

$$|Q_2^j \cap A| > \delta|Q_2^j|$$

and let $\mathcal{S}_2$ be the collection of such cubes. Continue the above process and let $\mathcal{S} = \bigcup_j \mathcal{S}_j$. Note that $\mathcal{S}$ is a countable collection of dyadic cubes. Write $\mathcal{S} = \{Q_k\}$. By construction, if $x \notin \bigcup_k Q_k$, then there exists a sequence of cubes $\{C_i(x)\}$ containing $x$ satisfying $|C_i(x)| \to 0$ as $i \to \infty$ and

$$|C_i(x) \cap A| < \delta|C_i(x)| < |C_i(x)|.$$

Hence it follows from the Lebesgue differentiation theorem that

$$\chi_A(x) = \lim_{i\to\infty} \frac{|C_i(x) \cap A|}{|C_i(x)|} < 1$$

for a.e. $x \in Q \setminus \bigcup_i Q_i$. This implies that $x \in Q \setminus A$. Hence

$$A \subset \bigcup_i Q_i$$

except for a set of measure zero, i.e., $|A \setminus \bigcup_k Q_k| = 0$. □

By construction, for each $Q_k$, there exists a finite nested sequence $\tilde{Q}_k^1$, ..., $\tilde{Q}_k^{r(k)}$ of dyadic cubes

$$\tilde{Q}_k^1 \supset \tilde{Q}_k^2 \supset \cdots \supset \tilde{Q}_k^{r(k)} \supset Q_k.$$

We call $\tilde{Q}_k^1$, ..., $\tilde{Q}_k^{r(k)}$ as the predecessor of cube $Q_k$.

Now we present a crawling an ink spot lemma. We will see the role of crawling an ink spot lemma to control the behavior of solutions of elliptic equations.

**Theorem 2.2** (Crawling an ink spot)**.** *Let $Q$ be a bounded cube in $\mathbb{R}^n$ and $A \subset B \subset Q$ satisfying $|A| < \delta|Q|$ for some $\delta \in (0,1)$. Assume further that for each dyadic cube $Q_k$ obtained from $Q$, $|A \cap Q_k| > \delta|Q_k|$ implies its predecessor $\tilde{Q}_k \subset B$. Then $|A| < \delta|B|$.*

*Remark.* (a) Interpretation (mine) of this lemma is that if $A$ is relatively small than $Q$ and if $B$ contains the predecessor of $Q_k$ for which the density of $A$ on $Q_k$ is greater than $\delta$, then at least the set $A$ has less information than $B$.

(b) According to Safonov, the growth lemma has an intuitive probabilistic interpretation, but the author does not know this. They used a refined covering lemma given by Herz and Stein instead of Calderón-Zygmund covering. See Safonov [22].

*Proof.* Let $\{Q_k\}$ be a covering given in Theorem 2.1. By assumption, for each $Q_k$ satisfying $|A \cap Q_k| > \delta|Q_k|$, we have $|A \cap \tilde{Q}_k| \leq \delta|\tilde{Q}_k|$ for any predecessor $\tilde{Q}_k$. By assumption, $\tilde{Q}_k \subset B$. Since

$$A \subset \bigcup_j Q_j \quad \text{except for a set of measure zero,}$$

it follows that

$$A \subset \bigcup_j \tilde{Q}_j \subset B \quad \text{except for a set of measure zero.}$$

Extract a disjoint subcovering and we relabel it by $\{\tilde{Q}_k\}$. Then

$$|A| \leq \sum_{k=1}^{\infty} |A \cap \tilde{Q}_k| \leq \delta|B|,$$

which completes the proof. $\square$

Although the above theorem is enough for this note, it is usually better to work on balls rather than cubes due to their geometric nature. We give a parabolic analog of Theorem 2.2. To introduce the theorem, for $r > 0$, we define the cylinder

$$Q_r(x,t) = B_r(x) \times (t - r^2, t]$$

and the time-shifted cylinder

$$\overline{Q}^m = B_r(x) \times (t, t + mr^2).$$

The following theorem is an analog of Theorem 2.2, which plays a crucial role in the regularity theory of parabolic equations, see e.g. Schwab-Silvestre [23] and Dong-Kim [7].

**Theorem 2.3.** *Let $E \subset F \subset B_{1/2} \times \mathbb{R}$. Assume that there exists a $\delta > 0$ such that*

(i) *For every point $(x,t) \in F$, there exists a cylinder $Q \subset B_1 \times \mathbb{R}$ so that $(x,t) \in Q$ and $|E \cap Q| \leq (1-\delta)|Q|$*
(ii) *For every cylinder $Q \subset B_1 \times \mathbb{R}$ such that $|E \cap Q| > (1-\delta)|Q|$, we have $\overline{Q}^m \subset F$.*

*Then there exists an absolute constant $c$ depending only on $n$ such that*

$$|E| \leq \frac{m+1}{m}(1 - c\delta)|F|.$$

Now we apply the crawling ink spot lemma to elliptic equations.

2.1. **Krylov-Safonov theorem.** In this subsection, we give a rough idea of obtaining the Krylov-Safonov theorem. In particular, we emphasize the role of the crawling ink spot lemma in the proof. The following theorem is due to Krylov-Safonov [18]. We note that they proved the parabolic version of this theorem.

**Theorem 2.4.** *Let $A = (a^{ij})$ be uniformly elliptic, i.e., there exist $\lambda, \Lambda > 0$ such that*
$$\lambda|\xi|^2 \leq a^{ij}(x)\xi_i\xi_j \leq \Lambda|\xi|^2 \quad \text{for all } x \in B_4 \quad \text{and} \quad \xi \in \mathbb{R}^n$$
*and let $u \in W^{2,n}(B_4)$ satisfy $\mathcal{L}u = a^{ij}D_{ij}u = 0$. Then there exists $0 < \alpha < 1$ depending only on $n$, $\lambda$, and $\Lambda$ such that $u \in C^\alpha(B_1)$.*

The strategy to get this theorem is

- (Local boundedness) If $u \in W^{2,n}(B_4)$ $\mathcal{L}u = 0$ in $B_4$, then for any $0 < p < \infty$, we have
$$\sup_{B_R(y)} |u| \leq C \left( \fint_{B_{2R}(y)} |u|^p \, dx \right)^{1/p}$$
for any ball $B_{2R}(y) \subset B_1$. This was achieved by Alexsandrov's maximum principle.
- If $u \in W^{2,n}(B_4)$ satisfy $u \geq 0$ and $\mathcal{L}u \leq 0$, then
$$\|u\|_{L^p(B_{1/2})} \leq C \inf_{B_{1/4}} u.$$
- (Harnack inequality) If $u \in W^{2,n}(B_4)$ satisfy $\mathcal{L}u \leq 0$ in $B_4$ and $u$ is non-negative in a ball $B_R(y) \subset B_1$, then
$$\sup_{B_{R/2}} u \leq C \inf_{B_{R/2}} u.$$
- (Hölder continuity) If $u \in W^{2,n}(B_4)$ satisfy $\mathcal{L}u = 0$ in $B_4$, then $u \in C^\alpha(B_1)$.

Here we apply the crawling an ink spot lemma to prove the second assertion. To explain it, we choose a cube $Q_1$ and $Q_3$ of side length 1 and 3, respectively. Then $B_{1/4} \subset B_{1/2} \subset Q_1 \subset Q_3 \subset B_{2\sqrt{n}}$. Then if there exist $\varepsilon_0 > 0$, $\mu \in (0,1)$, and $M > 1$ such that
$$\inf_{Q_3} u \leq 1$$
then
$$|\{u \leq M\} \cap Q_1| > \mu.$$
Moreover, we have
$$|\{u > M^k\} \cap Q_1| \leq (1-\mu)^k$$
for all $k = 1, 2, 3, \ldots$. The first part could be established by choosing the appropriate barrier function. Set
$$A = \{u > M^k\} \cap Q_1, \quad B = \{u > M^{k-1}\} \cap Q_1.$$
Then $A \subset B \subset Q_1$ and $|A| \leq 1 - \mu$. Suppose that if the following assertion is satisfied: if $Q_r(x_0)$ is a cube in $Q_1$ with $0 < r < 1/2$ satisfying
$$|A \cap Q_r(x_0)| > (1-\mu)|Q_r(x_0)|,$$
then $Q_{3r}(x_0) \cap Q_1 \subset B$. Then by the crawling an ink spot lemma, we conclude that $|A| \leq (1-\mu)|B|$.

For those who are interested in the proof of Krylov-Safonov, see Gilbarg-Trudinger [12] and Han [14].

## 3. John-Nirenberg theorem on BMO

The purpose of this section is twofold. We first define the space $BMO$, and we present proof of the John-Nirenberg theorem [16]. Next, we apply the John-Nirenberg theorem to the De Giorgi-Nash-Moser theorem.

### 3.1. **John-Nirenberg theorem.**

A locally integrable function $f : \mathbb{R}^n \to \mathbb{C}$ has a *bounded mean oscillation* if

$$\|f\|_{\mathrm{BMO}} = \sup_Q \frac{1}{|Q|} \int_Q |f(x) - f_Q| \, dx < \infty,$$

where the supremum is taken over all cubes $Q$ in $\mathbb{R}^n$ and $f_Q = \frac{1}{|Q|} \int_Q f \, dx$. It is easy to see that the cube in the definition can be replaced with balls. We say that $f \in \mathrm{BMO}$ if $f$ has bounded mean oscillation.

We list several properties of BMO.

- Bounded functions are in BMO.
- If $c > 0$ and $f \in \mathrm{BMO}$, then $cf \in \mathrm{BMO}$ and $\|cf\|_{\mathrm{BMO}} = c\|f\|_{\mathrm{BMO}}$.
- $W^{1,n}(\mathbb{R}^n)$ is continuously embedded into BMO.
- $\log|x|$ is $BMO$, but not $\log_+|x|$. Cutoffs are not BMO.
- There is a famous duality theorem due to Fefferman-Stein: the dual of $\mathcal{H}^1$ is BMO, where $\mathcal{H}^1$ denotes the Hardy space.
- If $T$ is a Calderón-Zygmund operator, then $T : \mathrm{BMO} \to \mathrm{BMO}$.

Recall the Chebyshev inequality: for any $\lambda > 0$ and $1 \le p < \infty$,

$$|\{x : |f(x) - a| > \lambda\}| \le \frac{\|f - a\|_{L^p}^p}{\lambda^p}.$$

The John-Nirenberg inequality gives similar information. It measures a distance of a function to a quantity.

**Theorem 3.1.** *Let $f \in \mathrm{BMO}$. Then there exist absolute constants $c_0 = e$ and $c_1 = 1/(2^n e)$ such that*

$$\frac{|\{x \in Q : |f(x) - f_Q| > \lambda\}|}{|Q|} \le c_0 e^{-c_1\lambda/\|f\|_{\mathrm{BMO}}} \tag{3.1}$$

*for all cube $Q$. In particular,*

$$⨍_Q \exp\left(\frac{|f(x) - f_Q|}{c_2\|f\|_{\mathrm{BMO}}}\right) dx \le c_2, \tag{3.2}$$

*where $c_2$ is a constant.*

*Proof.* By scaling, we may assume that $\|f\|_{\mathrm{BMO}} = 1$. Indeed, if we have (3.1) when $\|f\|_{\mathrm{BMO}} = 1$, then define $g = f/\|f\|_{\mathrm{BMO}}$. Then for $\lambda > 0$, it follows that

$$\left|\left\{x \in Q : |g(x) - g_Q| > \frac{\lambda}{\|f\|_{\mathrm{BMO}}}\right\}\right| \le c_0|Q| \exp\left(-\frac{c_1\lambda}{\|f\|_{\mathrm{BMO}}}\right).$$

Since

$$|g(x) - g_Q| = \frac{1}{\|f\|_{\mathrm{BMO}}}|f(x) - f_Q|,$$

we get the desired estimate.

To show the estimate, define

$$E(Q, \lambda) = \{x \in Q : |f(x) - f_Q| > \lambda\}$$

and

$$\varphi(\lambda) = \sup_Q \frac{|E(Q,\lambda)|}{|Q|}.$$

We will show that $\varphi(\lambda) \lesssim e^{-\lambda/c}$ for some absolute constant $c > 0$. Take $\lambda > e > 1$. Then by assumption,

$$\fint_Q |f - f_Q|\,dx \le \|f\|_{\mathrm{BMO}} = 1 < \lambda$$

for any cube $Q$.

We divide the cube $Q$ into $2^n$-congruent cubes. Let $\mathcal{G}_1$ be the collection of these cubes. Choose a cube $Q'$ in $\mathcal{G}_1$ satisfying

$$\fint_{Q'} |f - f_Q|\,dx > \lambda.$$

Let $\mathcal{S}_1$ be the collection of such cubes. For $Q' \in \mathcal{G}_1 \setminus \mathcal{S}_1$, decompose it into $2^n$-congruent cubes. Let $\mathcal{G}_2$ be the collection of these cubes. Choose a cube $Q'$ in $\mathcal{G}_2$ satisfying

$$\fint_{Q'} |f - f_Q|\,dx > \lambda.$$

Let $\mathcal{S}_2$ be the collection of such cubes. Continue this process and let $\mathcal{S} = \bigcup_j \mathcal{S}_j$. Since $\mathcal{S}$ is countable, we may write $\mathcal{S} = \{Q_j\}_{j=1}^{\infty}$. For $g(x) = |f(x) - f_Q|$, introduce the dyadic maximal operator

$$M_Q^d g(x) = \sup_{Q' \in \mathbb{D}_Q, x \in Q'} \fint_{Q'} g\,dx,$$

where $\mathbb{D}_Q$ is the collection of dyadic cubes obtained from $Q$. Note that

$$\{x \in Q : M_Q^d g(x) > \lambda\} = \bigcup_j Q_j.$$

For a.a. $x \in E(Q,\lambda)$, we have

$$\lambda < |f(x) - f_Q| = g(x) \le M_Q^d g(x).$$

Hence

$$E(Q,\lambda) \subset \bigcup_j Q_j \quad \text{for almost every } x. \tag{3.3}$$

Let $\hat{Q}_j$ be a parent cube of $Q_j$. Then

$$\lambda < \fint_{Q_j} |f - f_Q|\,dx \le \frac{|\hat{Q}_j|}{|Q_j|} \fint_{\hat{Q}_j} |f - f_Q|\,dx \le 2^n \lambda. \tag{3.4}$$

Now

$$\begin{aligned}
|f(x) - f_Q| &\le |f(x) - f_{Q_j}| + |f_{Q_j} - f_Q| \\
&\le |f(x) - f_{Q_j}| + \left| \fint_{Q_j} (f - f_Q)\,dx \right| \\
&\le |f(x) - f_{Q_j}| + \fint_{Q_j} |f - f_Q|\,dx \\
&\le |f(x) - f_{Q_j}| + 2^n \lambda.
\end{aligned}$$

For $x \in E(Q,t)$, $t > 2^n\lambda$, we have

$$t < |f(x) - f_Q| \le |f(x) - f_{Q_j}| + 2^n\lambda,$$

which implies that

$$|f(x) - f_{Q_j}| > t - 2^n\lambda.$$

Since $E(Q,\lambda) \supset E(Q,t)$, $t > \lambda$, it follows from (3.3) that

$$\begin{aligned}|E(Q,t)| &= |E(Q,t) \cap Q(t,\lambda)| \\ &\le \sum_j |E(Q,t) \cap Q_j| \\ &\le \sum_j \frac{|\{x \in Q_j : |f(x) - f_{Q_j}| > t - 2^n\lambda\}|}{|Q_j|}|Q_j| \\ &\le \varphi(t - 2^n\lambda)\sum_j |Q_j|.\end{aligned}$$

Hence by (3.4), we have

$$\begin{aligned}|E(Q,t)| &\le \varphi(t - 2^n\lambda)\frac{1}{\lambda}\sum_j \int_{Q_j} |f - f_Q|\, dx \\ &\le \frac{1}{\lambda}\varphi(t - 2^n\lambda)|Q|.\end{aligned}$$

By diving $|Q|$, and taking supremum over $Q$, we get

$$\varphi(t) \le \frac{1}{\lambda}\varphi(t - 2^n\lambda).$$

Put $\lambda = e$. Note that $\varphi(t) \le 1$ for all $t > 0$. So for $0 < t \le e \cdot 2^n$, we see that

$$\varphi(t) \le e \cdot e^{-\frac{t}{2^n e}}.$$

Note that

$$(0,\infty) = (0, e \cdot 2^n] \cup \bigcup_{k=1}^{\infty} (e \cdot 2^{n+k-1}, e \cdot 2^{n+k}].$$

Hence for $e \cdot 2^n < t < e \cdot 2^{n+1}$, we see that $\varphi(t) \le e \cdot e^{-t/(2^n e)}$. Since

$$\varphi(t) \le \frac{1}{e}\varphi(t - e \cdot 2^n), \quad t > e \cdot 2^n,$$

it follows that $\varphi(t - e \cdot 2^n) \le e \cdot e^{-(t - 2^n e)/2^n e)}$ for $t > e \cdot 2^n$. Hence for $e \cdot 2^n < t < e \cdot 2^{n+1}$, we have

$$\varphi(t) \le e \cdot e^{-t/(2^n e)}.$$

Continuing this process, then we obtain the desired result, which completes the proof of (3.1). To prove (3.2), we recall that

$$\int_X e^{|f|} - 1\, d\mu = \int_0^\infty e^\lambda \mu(\{x \in X : |f(x)| > \lambda\})\, d\lambda.$$

Hence for a measurable function $h$ on $\mathbb{R}^n$, we have

$$\begin{aligned}\fint_Q e^h\, dx &= 1 + \frac{1}{|Q|}\int_Q (e^h - 1)\, dx \\ &= 1 + \frac{1}{|Q|} e^\lambda \mu(\{x \in X : |f(x)| > \lambda\}) d\lambda.\end{aligned}$$

By taking $h = \gamma|f - f_Q|/\|f\|_{\mathrm{BMO}}$ with $\gamma < (2^n e)^{-1}$, we get

$$\fint_Q e^{\gamma|f(x)-f_Q|/\|f\|_{\mathrm{BMO}}}\,dx \le \int_0^\infty e^{\lambda} e e^{-A(\lambda\|f\|_{\mathrm{BMO}})/\gamma} d\lambda = C_{n,\gamma}.$$

This completes the proof. $\square$

3.2. **Application of John-Nirenberg theorem.** Here we apply the John-Nirenberg theorem to the regularity theory of elliptic equations.

**Theorem 3.2** (De Giorgi-Nash-Moser). *Let $A = (a^{ij})$ be uniformly elliptic, i.e., there exist $\lambda, \Lambda > 0$ such that*

$$\lambda|\xi|^2 \le a^{ij}(x)\xi_i\xi_j \le \Lambda|\xi|^2 \quad \text{for all } x \in B_1 \quad \text{and} \quad \xi \in \mathbb{R}^n.$$

*If $u \in W^{1,2}(B_1)$ is a weak solution of $-\operatorname{div}(A\nabla u) = 0$ in $B_1$, i.e., $u$ satisfies*

$$\int_{B_1} a^{ij} D_j u D_i \varphi\, dx = 0$$

*for all $\varphi \in C_c^\infty(B_1)$, then $u \in C^\alpha_{loc}(B_1)$ for some $0 < \alpha < 1$.*

Although there are several approaches to show this theorem, we briefly explain Moser's method to connect the John-Nirenberg inequality with the regularity theory of elliptic equations.

The strategy of Moser is the following.

- We first show that every nonnegative weak subsolution is locally bounded in $B_{1/2}$
$$\sup_{B_r} u \le C\frac{1}{(R-r)^{n/p}}\|u^+\|_{L^p(B_R)}$$
for any $0 < p < \infty$.
- If $u$ is a nonnegative supersolution in $B_1$, i.e.,
$$\int_{B_1} a_{ij} D_i u D_j \varphi\, dx \ge 0 \quad \text{for all } \phi \in W_0^{1,2}(B_1) \quad \text{and } \varphi \ge 0,$$
then for any $0 < \theta < \tau < 1$ and $B_R(x) \subset B_1$,
$$\inf_{B_{\theta R}(x)} u \ge C\left(\frac{1}{R^n}\int_{B_{\tau R}} u^p\, dx\right)^{1/p}$$
for any $0 < p < n/(n-2)$.
- Combining these two estimates, then any nonnegative weak solution in $B_1$ satisfies
$$\sup_{B_{R/4}(x)} u \le C \inf_{B_{R/4}(x)} u$$
for any $B_R(x) \subset B_1$. This inequality is called *Harnack's inequality*
- As a consequence of Harnack's inequality, we obtain the $C^\alpha$-regularity.

If we define

$$\Phi(p, r) = \left(\int_r |\overline{u}|^p\, dx\right)^{1/p},$$

where $\overline{u} = u + k$, then we could obtain

$$\Phi(\chi\gamma, r_1) \le \left(\frac{C(1+|\gamma|)^{\sigma+1}}{r_2 - r_1}\right)^{2/|\gamma|} \Phi(\gamma, r_2) \quad \text{if } \gamma > 0$$

$$\Phi(\gamma, r_2) \le \left(\frac{C(1+|\gamma|)^{\sigma+1}}{r_2 - r_1}\right)^{2/|\gamma|} \Phi(\chi\gamma, r_2) \quad \text{if } \gamma < 0$$

for $1 \le r_1 < r_2 \le 3$ and $\chi = \hat{n}/(\hat{n}-2)$. Here $\hat{n} = n$ if $n \ge 3$ and $\hat{n} > 2$ if $n = 2$. Then by taking successive iteration, one could obtain

$$\Phi(\chi^m p, 1) \le C\Phi(p, 2),$$

which implies that

$$\sup_{B_r} u \le C\Phi(p, 2).$$

Similarly, for any $0 < p_0 < p < \chi$ and $\gamma < 1$, we obtain

$$\Phi(p, 2) \le C\Phi(p_0, 3), \quad \Phi(-p_0, 3) \le C \inf_{B_r} \overline{u}, \tag{3.5}$$

Hence, we need to show that $\Phi(p_0, 3) \le C\Phi(-p_0, 3)$ for some $C > 0$. The key step is to prove the following inequality: for any $0 < \tau < 1$, there exists a constant $C = C(n, \lambda, \Lambda, \tau)$ such that

$$\int_{B_\tau} e^{p_0|w|}\, dx \le C, \tag{3.6}$$

where $\overline{u} = \log(u + k) - \beta$ $w = \overline{u} - \beta > 0$ for some $k > 0$ and $\beta = ⨍_{B_\tau} \log \overline{u}$. By definition of $w$, this leads us to prove the desired estimate (3.5).

It can be shown that

$$\int_{B_1} |\nabla w|^2 \zeta^2\, dx \le C \int_{B_1} |\nabla \zeta|^2\, dx$$

for all $\zeta \in C_c^1(B_1)$. Then for any $B_{2r}(y) \subset B_1$, choose $\zeta$ satisfying

$$\operatorname{supp} \zeta \subset B_{2r}(y), \quad \zeta = 1 \quad \text{in } B_r(y), \quad |\nabla\zeta| \le \frac{2}{r}.$$

Hence

$$\int_{B_r(y)} |\nabla w|^2\, dx \le C r^{n-2}.$$

By the Poincaré inequality, we have

$$\begin{aligned} \frac{1}{r^n} \int_{B_r(y)} |w - w_{B_r(y)}|\, dx &\le \frac{1}{r^{n/2}} \left( \int_{B_r(y)} |w - w_{B_r(y)}|\, dx \right)^{1/2} \\ &\le \frac{c}{r^{n/2}} \left( r^2 \int_{B_r(y)} |\nabla w|^2\, dx \right)^{1/2} \le C. \end{aligned}$$

Hence it follows that $w \in \mathrm{BMO}$. Therefore, it follows from Theorem 3.1 that (3.6) holds.

For a detailed explanation of this topic, see Giaquinta-Martinazzi [11], Gilbarg-Trudinger [12], and Han-Lin [15].

## 4. Gehring lemma

We proved local Hölder continuity of solutions to elliptic equations in divergence form and nondivergence form. We note that we proved local Hölder continuity when these are scalar equations. So one might ask whether we do have a local regularity result for elliptic systems. However, the following counterexample due to De Giorgi [5] suggests that there exists an elliptic system whose solution may not bounded when $n \geq 3$.

**Example 4.1.** Let

$$A_{ij}^{\alpha\beta}(x) = \delta_{\alpha\beta}\delta_{ij} + \left[(n-2)\delta_{\alpha i} + n\frac{x_i x_\alpha}{|x|^2}\right]\left[(n-2)\delta_{\beta j} + n\frac{x_j x_\beta}{|x|^2}\right].$$

The coefficient is bounded and satisfies Legendre condition. Define

$$u(x) := \frac{x}{|x|^\gamma}, \quad \gamma = \frac{n}{2}\left[1 - ((2n-2)^2+1)^{-1/2}\right],$$

which belongs to $W^{1,2}(B_1)$, but it is not bounded.

Nevertheless, we do have another type of regularity result, higher integrability which can be established by the Reverse Hölder estimate. Reverse Hölder's inequality appears in several places in modern analysis. An interesting consequence of reverse Hölder inequality is that it gives a self-improving property of the integrability. The first related result is due to Gehring [9] in the connection with the theory of quasiconformal mappings. He proved that if $g$ is a nonnegative function defined on a cube $Q$ in $\mathbb{R}^n$ and it is zero outside of $Q$ and $g$ satisfies

$$M(g^q) \leq bM(g)^q$$

for some constant $b > 1$, then $g \in L^p(Q)$ for $q \leq p < q + \varepsilon$ and

$$\left(\fint_Q g^p\,dx\right)^{1/p} \leq c\left(\fint_Q g^q\,dx\right)^{1/q}$$

where $\varepsilon$ and $c$ are positive constants depending only on $q$, $b$, and $n$. Here $Mf$ denotes the classical Hardy-Littlewood maximal function.

Unfortunately, the above result is not suitable to many situation. As an example, if $u$ is a weak solution of

$$\triangle u = 0 \quad \text{in } \Omega$$

then for $B_R \Subset \Omega$, we have the following Caccioppoli inequality:

$$\int_{B_{R/2}} |\nabla u|^2\,dx \leq \frac{c}{R^2}\int_{B_R} |u - u_{B_R}|^2\,dx.$$

Hence by Poincaré-Sobolev's inequality, we get

$$\left(\fint_{B_{R/2}} |\nabla u|^2\,dx\right)^{1/2} \leq c\left(\fint_{B_R} |\nabla u|^q\,dx\right)^{1/q}$$

where $q = \frac{2n}{n+2} < 2$. Observe that the integrand of the left-hand side is smaller than that of the right-hand side.

Motivated by this, a local version of Gehring's result was shown by Giaquinta-Modica [10] to develop regularity theory for nonlinear elliptic equations. They proved a local reverse Hölder inequality by using more refined covering argument. For the parabolic extension, see Kinnunen-Lewis [17] for details. For the application of reverse Hölder's inequality to Navier-Stokes equation, see e.g. Choe-Lewis [4] and Choe-Yang [3].

Here we state the self-improving property of reverse Hölder inequality.

**Theorem 4.2.** *Fix a cube $Q_0$ in $\mathbb{R}^n$. Let $g \geq 0$, $f \geq 0$, $g \in L^q(Q_0)$, $f \in L^r(Q_0)$, $1 < q < r$. There exists a constant $0 < \theta = \theta(n,q) < 1$ such that if there exists $b > 0$ such that*

$$\fint_{Q_{R/2}} g^q\,dx \leq b\left[\left(\fint_{Q_R} g\,dx\right)^q + \fint_{Q_R} f^q\,dx\right] + \theta\fint_{Q_R} g^q\,dx \tag{4.1}$$

*for each $Q_R = Q_R(x_0)$ contained in $Q_0$. then there exists $q \leq p < q < r$ such that there exists a constant $c = c(q,n,\theta,b) > 0$ such that*

$$\left(\fint_{Q_{R/2}} g^p\,dx\right)^{1/p} \leq c_p\left[\left(\fint_{Q_R} g^q\,dx\right)^{1/q} + \left(\fint_{Q_R} f^p\,dx\right)^{1/p}\right]$$

*for each $Q_R = Q_R(x_0)$ contained in $Q_0$.*

See [10] or [11] for the proof.

Finally, we give an application of the reverse Hölder estimate to the regularity theory for linear elliptic systems

$$-\operatorname{div}(A\nabla u) = 0 \quad \text{in } \Omega.$$

**Theorem 4.3.** *Suppose that $A_{ij}^{\alpha\beta}(x)$ satisfy*

$$A_{ij}^{\alpha\beta}(x)\xi_\alpha^i\xi_\beta^j \geq \lambda|\xi|^2, \quad \lambda > 0; \quad A_{ij}^{\alpha\beta} \in L^\infty,\ |A_{ij}^{\alpha\beta}| \leq \Lambda;$$

*and $u \in W^{1,2}(\Omega;\mathbb{R}^N)$ satisfies*

$$\int_\Omega A_{ij}^{\alpha\beta}(x)D_\beta u^j D_\alpha\varphi^i\,dx = 0, \quad \textit{for all } \varphi \in W_0^{1,2}(\Omega;\mathbb{R}^N).$$

*Here we follow the Einstein summation convention. Then there exists $p > 2$ such that $|\nabla u| \in L^p_{\mathrm{loc}}(\Omega)$, and for $B_R \subset \Omega$, we have*

$$\left(\fint_{B_{R/2}} |\nabla u|^p\,dx\right)^{1/p} \leq C\left(\fint_{B_R} |\nabla u|^2\,dx\right)^{1/2}$$

*where $C$ and $p$ depend on $n$, $N$, $\lambda$, $\Lambda$.*

*Proof.* Fix $B \subset \Omega$ and $x_0 \in B$. Then for $0 < R < \operatorname{dist}(x_0, \partial B)$, we first derive the following Caccioppoli inequality:

$$\int_{B_{R/2}(x_0)} |\nabla u|^2\,dx \leq \frac{C}{R^2}\int_{B_R(x_0)} |u - u_R|^2\,dx,$$

where $C = C(n,N,\lambda,\Lambda)$ and $u_R = \fint_{B_R} u\,dx$.

Choose a test function $\zeta \in C_c^\infty(B_R(x_0))$ so that $0 \leq \zeta \leq 1$, $\zeta = 1$ on $B_{R/2}(x_0)$, and $|\nabla\zeta| \leq C/R$. Put $\varphi = \zeta^2(u - u_R)$. Then $\varphi \in W_0^{1,2}(\Omega;\mathbb{R}^n)$. Then by product rules, we have

$$\begin{aligned} 0 = \int_\Omega A_{ij}^{\alpha\beta} D_\beta u^j D_\alpha(\zeta^2 u^i)\,dx = &\int_\Omega A_{ij}^{\alpha\beta}(D_\beta u^j)(D_\alpha u^i)\zeta^2\,dx \\ &+2\int_\Omega A_{ij}^{\alpha\beta} D_\beta u^j 2\zeta(D_\alpha\zeta)(u^i - u_R^i)\,dx. \end{aligned}$$

By ellipticity, we have

$$A_{ij}^{\alpha\beta}(D_\beta u^j)(D_\alpha u^i) \geq \lambda|\nabla u|^2.$$

Hence we get

$$\lambda \int_{B_R(x_0)} \zeta^2 |\nabla u|^2\, dx \leq \Lambda \int_\Omega |\nabla u||u - u_{B_R}||2\zeta||\nabla\zeta|\, dx.$$

Then by Cauchy-Schwarz inequality and Young's inequality, we get

$$\lambda \int_{B_R(x_0)} \zeta^2 |\nabla u|^2\, dx \leq \frac{1}{2}\int_{B_R(x_0)} \zeta^2|\nabla u|^2\, dx + \frac{C}{R^2}\int_{B_R(x_0)} |u - u_{B_R}|^2\, dx,$$

which implies the desired assertion.

By Poincaré-Sobolev inequality, we have

$$\left(\int_{B_R(x_0)} |u - u_R|^2\, dx\right)^{1/2} \leq C(n)\left(\int_{B_R(x_0)} |\nabla u|^{2n/(n+2)}\, dx\right)^{(n+2)/(2n)}.$$

Hence we get

$$\fint_{B_{R/2}(x_0)} |\nabla u|^2\, dx \leq C\left(\fint_{B_R(x_0)} |\nabla u|^{2n/(n+2)}\, dx\right)^{(n+2)/n},$$

where $C = C(n, N, \lambda, \Lambda)$. Since $2 > 2n/(n+2)$, we can apply Theorem 4.2 to $g = |\nabla u|^{2n/(n+2)}$, $q = (n+2)/n$, $\theta = 0$, $f = 0$. Then we have

$$|\nabla u|^{2n/(n+2)} \in L^r_{\mathrm{loc}}(B), \quad \frac{n+2}{n} \leq r < \frac{n+2}{n} + \varepsilon,$$

and for any $B_R \subset B$, we have the estimate

$$\left(\fint_{B_{R/2}} |\nabla u|^{2nr/(n+2)}\, dx\right)^{1/r} \leq C\left(\fint_{B_R} |\nabla u|^2\, dx\right)^{n/(n+2)}.$$

Set $p = 2nr/(n+2)$. Then $p > 2$ and

$$\left(\fint_{B_{R/2}} |\nabla u|^p\, dx\right)^{1/p} \leq C\left(\fint_{B_R} |\nabla u|^2\, dx\right)^{1/2}$$

where $C = C(n, N, \lambda, \Lambda)$. This completes the proof of Theorem 4.3. □

**Example 4.4.** Here the exponent $p$ cannot be arbitrarily large. The following example is due to Meyer [19]. Let $N = 1$, $n = 2$, $\Omega = B_1$. Consider

$$(au_x + bu_y)_x + (bu_x + cu_y)_y = 0, \quad (x, y) \in B_1, \tag{4.2}$$

where

$$a = 1 - (1-\mu^2)\frac{y^2}{x^2+y^2},$$
$$b = (1-\mu^2)\frac{xy}{x^2+y^2},$$
$$c = 1 - (1-\mu^2)\frac{x^2}{x^2+y^2},$$

where $0 < \mu < 1$ is a fixed constant. Then the matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ has eigenvalue $\mu^2$ and 1. Also,

$$u(x, y) = x(x^2 + y^2)^{(\mu-1)/2}$$

is a weak solution of (4.2), and

$$|\nabla u| \in L^p_{\mathrm{loc}}(B_1), \quad 2 \le p < \frac{2}{1-\mu}.$$

Since $\int_{B_1} |\nabla u|^{2/(1-\mu)}\, dxdy = \infty$, $|\nabla u| \in L^p_{\mathrm{loc}}(B_1)$ only if $p < 2/(1-\mu)$.

## 5. Further results

There is another application of Calderón-Zygmund decomposition. See Caffarelli-Peral [2] and Byun [1].

## References


1. Sun-Sig Byun, *Elliptic equations with BMO coefficients in Lipschitz domains*, Trans. Amer. Math. Soc. **357** (2005), no. 3, 1025–1046. MR 2110431
2. L. A. Caffarelli and I. Peral, *On $W^{1,p}$ estimates for elliptic equations in divergence form*, Comm. Pure Appl. Math. **51** (1998), no. 1, 1–21. MR 1486629
3. Hi Jun Choe and Minsuk Yang, *Local kinetic energy and singularities of the incompressible Navier-Stokes equations*, J. Differential Equations **264** (2018), no. 2, 1171–1191. MR 3720841
4. ______, *The Minkowski dimension of boundary singular points in the Navier-Stokes equations*, J. Differential Equations **267** (2019), no. 8, 4705–4718. MR 3983051
5. Ennio De Giorgi, *Sulla differenziabilità e l'analiticità delle estremali degli integrali multipli regolari*, Mem. Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. (3) **3** (1957), 25–43. MR 0093649
6. ______, *Un esempio di estremali discontinue per un problema variazionale di tipo ellittico*, Boll. Un. Mat. Ital. (4) **1** (1968), 135–137. MR 0227827
7. Hongjie Dong and Doyoon Kim, *$L_p$-estimates for time fractional parabolic equations with coefficients measurable in time*, Adv. Math. **345** (2019), 289–345. MR 3899965
8. Javier Duoandikoetxea, *Fourier analysis*, Graduate Studies in Mathematics, vol. 29, American Mathematical Society, Providence, RI, 2001, Translated and revised from the 1995 Spanish original by David Cruz-Uribe. MR 1800316
9. F. W. Gehring, *The $L^p$-integrability of the partial derivatives of a quasiconformal mapping*, Acta Math. **130** (1973), 265–277. MR 402038
10. M. Giaquinta and G. Modica, *Regularity results for some classes of higher order nonlinear elliptic systems*, J. Reine Angew. Math. **311(312)** (1979), 145–169. MR 549962
11. Mariano Giaquinta and Luca Martinazzi, *An introduction to the regularity theory for elliptic systems, harmonic maps and minimal graphs*, second ed., Appunti. Scuola Normale Superiore di Pisa (Nuova Serie) [Lecture Notes. Scuola Normale Superiore di Pisa (New Series)], vol. 11, Edizioni della Normale, Pisa, 2012. MR 3099262

12. David Gilbarg and Neil S. Trudinger, *Elliptic partial differential equations of second order*, Classics in Mathematics, Springer-Verlag, Berlin, 2001, Reprint of the 1998 edition. MR 1814364
13. Loukas Grafakos, *Modern Fourier analysis*, third ed., Graduate Texts in Mathematics, vol. 250, Springer, New York, 2014. MR 3243741
14. Qing Han, *Nonlinear elliptic equations of the second order*, Graduate Studies in Mathematics, vol. 171, American Mathematical Society, Providence, RI, 2016. MR 3468839
15. Qing Han and Fanghua Lin, *Elliptic partial differential equations*, second ed., Courant Lecture Notes in Mathematics, vol. 1, Courant Institute of Mathematical Sciences, New York; American Mathematical Society, Providence, RI, 2011. MR 2777537
16. F. John and L. Nirenberg, *On functions of bounded mean oscillation*, Comm. Pure Appl. Math. **14** (1961), 415–426. MR 131498
17. Juha Kinnunen and John L. Lewis, *Higher integrability for parabolic systems of $p$-Laplacian type*, Duke Math. J. **102** (2000), no. 2, 253–271. MR 1749438
18. N. V. Krylov and M. V. Safonov, *A property of the solutions of parabolic equations with measurable coefficients*, Izv. Akad. Nauk SSSR Ser. Mat. **44** (1980), no. 1, 161–175, 239. MR 563790
19. Norman G. Meyers, *An $L^p$ e-estimate for the gradient of solutions of second order elliptic divergence equations*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (3) **17** (1963), 189–206. MR 159110
20. Jürgen Moser, *On Harnack's theorem for elliptic differential equations*, Comm. Pure Appl. Math. **14** (1961), 577–591. MR 159138
21. J. Nash, *Continuity of solutions of parabolic and elliptic equations*, Amer. J. Math. **80** (1958), 931–954. MR 100158
22. M. V. Safonov, *Harnack's inequality for elliptic equations and Hölder property of their solutions*, Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) **96** (1980), 272–287, 312, Boundary value problems of mathematical physics and related questions in the theory of functions, 12. MR 579490
23. Russell W. Schwab and Luis Silvestre, *Regularity for parabolic integro-differential equations with very irregular kernels*, Anal. PDE **9** (2016), no. 3, 727–772. MR 3518535
24. Elias M. Stein, *Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals*, Princeton Mathematical Series, vol. 43, Princeton University Press, Princeton, NJ, 1993, With the assistance of Timothy S. Murphy, Monographs in Harmonic Analysis, III. MR 1232192

Division of Applied Mathematics, Brown University, 182 George Street, Providence, RI 02912, USA

*Email address*: `hyunwoo_kwon@brown.edu`